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Loop quantum gravity and Planck-size black hole entropy

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Abstract

The Loop Quantum Gravity (LQG) program is briefly reviewed and one of its main applications, namely the counting of black hole entropy within the framework is considered. In particular, recent results for Planck size black holes are reviewed. These results are consistent with an asymptotic linear relation (that fixes uniquely a free parameter of the theory) and a logarithmic correction with a coefficient equal to $-1/2$. The account is tailored as an introduction to the subject for non-experts.

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I. INTRODUCTION

Loop Quantum Gravity (LQG) has become in the past years a mayor player as a candidate for a quantum theory of gravity [1, 2]. On the one hand it has matured into a serious contender together with other approaches such as *String/M Theory*, but on the other is it not as well understood, neither properly credited as a real physical theory of quantum gravity. The purpose of this contribution is twofold. First, we would like to provide a starting point for those interested in learning the basics of the theory and to provide at the same time an introduction to one of its application, namely the treatment of black holes and its entropy counting.

Loop quantum gravity is based and two basic principles, namely the general principles of quantum theory and the lessons from general relativity: that physics is diffeomorphism invariant. This means that the field describing the gravitational interaction, and the geometry of spacetime is fully dynamical and interacting with the rest of the fields present; when one considers the description of the full gravity-matter system, this better be a background independent one. The fact that LQG is based in general principles of quantum mechanics means only that one is looking for a description based on the language of Quantum Mechanics: states are elements on a Hilbert space (well defined, one expects), observables will be Hermitian operators thereon, etc. This does not mean that one should use *all* that is already known about quantizing fields. Quite on the contrary, the tools needed to construct a background independent quantization (certainly not like the quantization we know), are rather new.

Another important feature of LQG is that it is the most serious attempt to perform a full *non-perturbative* quantization of the gravitational field by itself. It is an attempt to answer the following question: can we quantize the gravitational degrees of freedom without considering matter on the first place? Since LQG aims at being a physical theory, which means it better be falsifiable, one expects to answer that question unambiguously, whenever one has the theory fully developed. This is one of the main present challenges of the theory, namely to produce predictions that can be tested experimentally. On the other hand, one can consider situations in which the full knowledge of the quantum theory is not needed in order to describe a particular physical situation. This is precisely the case of black hole horizons and their entropy.

Black hole entropy is one of the most intriguing constructs of modern theoretical physics. On the one hand, it has a correspondence with the black hole horizon area through the laws of (classical) black hole mechanics [3]. On the other hand it is assumed to have a quantum statistical origin given that the proper identification between entropy and area $S = A/4\ell_p^2$ came only after an analysis of *quantum* fields on a fixed background [4]. One of the most crucial test that a candidate quantum theory of gravity must pass then is to provide a mechanism to account for the microscopic degrees of freedom of black holes. It is not unfair to say that at the moment we have only two candidates for quantum gravity that have offered such an explanation: string/brane theory [5] and loop quantum gravity [6]. The LQG formalism uses as starting point isolated horizon (IH) boundary conditions at the classical level, where the interior of the BH is excluded from the region under consideration. In this sense, the description is somewhat effective, since part of the information about the interior is encoded in the boundary conditions at the horizon that in the quantum theory get promoted to a condition that horizon states must satisfy. There is also an important issue regarding this formalism. Loop quantum gravity possesses a one parameter family of

inequivalent representations of the classical theory labelled by a real number γ , the so called Barbero-Immirzi (BI) parameter (it is the analogue of the θ ambiguity in QCD [7]). It turns out that the BH entropy calculation provides a linear relation between entropy and area for very large black holes (in Planck units) as,

$$S = \lambda A(\gamma),$$

where the parameter λ is independent of γ and depends only in the counting. We have put the γ dependence in the Area, since the parameter appears explicitly in the area spectrum. The strategy that has been followed within the LQG community is to regard the Bekenstein-Hawking entropy $S = A/4$ as a requirement that large black holes should satisfy. This fixes uniquely the value of $\gamma = \gamma_0$ once and for all, by looking at the asymptotic behavior, provided that one has the ‘correct counting’ that provides the right value for γ_0 . The analytic counting has provided also an expression for the first correction term that turns out to be logarithmic [8, 9].

The purpose of the present article is to provide an introduction to the main ideas behind the loop quantum gravity program and to one of its main applications, namely the computation of black hole entropy. In this regard, we shall describe some recent results that have been performed for *Planck size* black holes and that complement in a precise way the analytical computations. In particular, as we shall show, even when the black holes considered are outside the original domain of applicability of the formalism, one can still learn from these considerations.

The structure of the paper is as follows. In Sec. II we present some preliminaries, such as the standard geometrodynamical variables for canonical gravity, the passage to connections and triads and the choice of classical observables to be quantized. In Sec. III we describe the loop quantum geometry formalism, including some relevant geometric operators. Sec. IV is devoted to the formalism of quantum isolated horizons. We recall the classical formalism and the basic steps to the quantization of the horizon theory. State counting and black hole entropy is the subject of Sec. V. We end with a discussion in Sec. VI.

II. PRELIMINARIES

A. Geometrodynamics

The first step is to introduce the basic classical variables of the theory. Since the theory is described by a Hamiltonian formalism, this means that the 4-dim spacetime M of the form $M = \Sigma \times \mathbb{R}$, where Σ is a 3-dimensional manifold. The first thing to do is to start with the geometrodynamical phase space Γ_g of Riemannian metrics q_{ab} on Σ and their canonical momenta $\tilde{\pi}^{ab}$ (related to the extrinsic curvature K_{ab} of Σ into M by $\tilde{\pi}^{ab} = \sqrt{q}(K^{ab} - \frac{1}{2}q^{ab}K)$, with $q = \det(q_{ab})$ and $K = q^{ab}K_{ab}$). Recall that they satisfy,

$$\{\tilde{\pi}^{ab}(x), q_{cd}(y)\} = 2\kappa \delta_{(c}^a \delta_{d)}^b \delta^3(x, y) \quad ; \quad \{q_{ab}(x), q_{cd}(y)\} = \{\tilde{\pi}^{ab}(x), \tilde{\pi}^{cd}(y)\} = 0 \quad (1)$$

General Relativity in these geometrodynamical variables is a theory with constraints, which means that the canonical variables $(q_{ab}, \tilde{\pi}^{ab})$ do not take arbitrary values but must satisfy four constraints:

$$\mathcal{H}^b = D_a(\tilde{\pi}^{ab}) \approx 0 \quad \text{and,} \quad \mathcal{H} = \sqrt{q} [R^{(3)} + q^{-1}(\frac{1}{2}\tilde{\pi}^2 - \tilde{\pi}^{ab}\tilde{\pi}_{ab})] \approx 0 \quad (2)$$

The first set of constraints are known as the vector constraint and what they generate (its gauge orbit) are spatial diffeomorphisms on Σ . The other constraint, the scalar constraint (or super-Hamiltonian) generates “time reparametrizations”. We start with 12 degrees of freedom, minus 4 constraints means that the constraint surface has 8 dimensions (per point) minus the four gauge orbits generated by the constraints giving the four phase space degrees of freedom, which corresponds to the two polarizations of the gravitational field.

B. Connection Dynamics

In order to arrive at the connection formulation, we need first to enlarge the phase space Γ_g by considering not metrics q_{ab} but the co-triads e_a^i that define the metric by,

$$q_{ab} = e_a^i e_b^j \delta_{ij} \quad (3)$$

where $i, j = 1, 2, 3$ are internal labels for the frames. These represent 9 variables instead of the 6 defining the metric q_{ab} , so we have introduced more variables, but at the same time a new symmetry in the theory, namely the $SO(3)$ rotations in the triads. Recall that a triad e_a^i and a rotated triad $e_a'^i(x) = U^i_j(x) e_a^j(x)$ define the same metric $q_{ab}(x)$, with $U^i_j(x) \in SO(3)$ a local rotation. In order to account for the extra symmetry, there will be extra constraints (first class) that will get rid of the extra degrees of freedom introduced. Let us now introduce the densitized triad as follows:

$$\tilde{E}_i^a = \frac{1}{2} \epsilon_{ijk} \tilde{\eta}^{abc} e_b^j e_c^k \quad (4)$$

where η^{abc} is the naturally defined levi-civita density one antisymmetric object. Note that $\tilde{E}_i^a \tilde{E}_j^b \delta^{ij} = q q^{ab}$.

Let us now consider the canonical variables. It turns out that the canonical momenta to the densitized triad \tilde{E}_i^a is closely related to the extrinsic curvature of the metric,

$$K_a^i = \frac{1}{\sqrt{\det(\tilde{E})}} \delta^{ij} \tilde{E}_j^b K_{ab} \quad (5)$$

For details see [14]. Once one has enlarged the phase space from the pairs $(q_{ab}, \tilde{\pi}^{ab})$ to (\tilde{E}_i^a, K_b^j) , the next step is to perform the canonical transformation to go to the *Ashtekar – Barbero* variables. First we need to introduce the so called *spin connection* Γ_a^i , the one defined by the derivative operator that annihilates the triad e_a^i (in complete analogy to the Christoffel symbol that defined the covariant derivative D_a killing the metric). It can be inverted from the form,

$$\partial_{[a} e_{b]}^i + \epsilon^i_{jk} \Gamma_a^j e_b^k = 0 \quad (6)$$

This can be seen as an extension of the covariant derivative to objects with mixed indices. The key to the definition of the new variables is to combine these two objects, namely the spin connection Γ with the object K_a^i (a tensorial object), to produce a new connection

$$\gamma A_a^i := \Gamma_a^i + \gamma K_a^i \quad (7)$$

This is the *Ashtekar-Barbero Connection*. Similarly, the other conjugate variable will be the rescaled triad,

$$\gamma \tilde{E}_i^a = \tilde{E}_i^a / \gamma \quad (8)$$

Now, the pair $(\gamma A_a^i, \gamma \tilde{E}_i^a)$ will coordinatize the new phase space Γ_γ . We have emphasized the parameter γ since this labels a one parameter family of different classically equivalent theories, one for each value of γ . The real and positive parameter γ is known as the Barbero-Immirzi parameter [10, 11]. In terms of these new variables, the canonical Poisson brackets are given by,

$$\{\gamma A_a^i(x), \gamma \tilde{E}_j^b(y)\} = \kappa \delta_a^b \delta_j^i \delta^3(x, y). \quad (9)$$

and,

$$\{\gamma A_a^i(x), \gamma A_b^j(y)\} = \{\gamma \tilde{E}_i^a(x), \gamma \tilde{E}_j^b(y)\} = 0 \quad (10)$$

The new constraint that arises because of the introduction of new degrees of freedom takes a very simple form,

$$G_i = \mathcal{D}_a \tilde{E}_i^a \approx 0 \quad (11)$$

that is, it has the structure of Gauss' law in Yang-Mills theory. We have denoted by \mathcal{D} the covariant defined by the connection γA_a^i , such that $\mathcal{D}_a \tilde{E}_i^a = \partial_a \tilde{E}_i^a + \epsilon_{ij}^{k} \gamma A_a^j \tilde{E}_k^a$. The vector and scalar constraints now take the form,

$$V_a = F_{ab}^i \tilde{E}_i^b - (1 + \gamma^2) K_a^i G_i \approx 0 \quad (12)$$

where $F_{ab}^i = \partial_a \gamma A_b^i - \partial_b \gamma A_a^i + \epsilon_{ij}^{k} \gamma A_a^j \gamma A_b^k$ is the curvature of the connection γA_b^j . The other constraint is,

$$\mathcal{S} = \frac{\tilde{E}_i^a \tilde{E}_j^b}{\sqrt{\det(\tilde{E})}} \left[\epsilon^{ij}{}_{k} F_{ab}^k - 2(1 + \gamma^2) K_{[a}^i K_{b]}^j \right] \approx 0 \quad (13)$$

The next step is to consider the right choice of variables, now seen as functions of the phase space Γ_γ that are preferred for the non-perturbative quantization we are seeking. As we shall see, the guiding principle will be that the functions (defined by an appropriate choice of smearing functions) will be those that can be defined without the need of a background structure, i.e. a metric on Σ .

C. Holonomies and Fluxes

Since the theory possesses these constraints, the strategy to be followed is to quantize first and then to impose the set of constraints as operators on a Hilbert space. This is known as the Kinematical Hilbert Space \mathcal{H}_{kin} . One of the main achievements of LQG is that this space has been rigourously defined.

Let us start by considering the connection A_a^i . The most natural object one can construct from a connection is a holonomy $h_\alpha(A)$ along a loop α . This is an element of the gauge group $G = SU(2)$ and is denoted by,

$$h_\alpha(A) = \mathcal{P} \exp \left(\oint_\alpha A_a ds^a \right) \quad (14)$$

The path-order exponential of the connection. Note that for notational simplicity we have omitted the 'lie-algebra indices'. From the holonomy, it is immediate to construct a gauge invariant function by taking the trace arriving then at the Wilson loop $T[\alpha] := \frac{1}{2} \text{Tr} \mathcal{P} \exp (\oint_\alpha A_a ds^a)$.

In recent years the emphasis has shifted from loops to consider instead closed graphs Υ , that consist of N edges e_I ($I = 1, 2, \dots, N$), and M vertices v_μ , with the restriction that there are no edges with ‘loose ends’. Given a graph Υ , one can consider the parallel transport along the edges e_I , the end result is an element of the gauge group $g_I = h(e_I) \in G$ for each such edge. One can then think of the connection A_a^i as a map from graphs to N -copies of the gauge group: $A_a^i : \Upsilon \rightarrow G^N$. Furthermore, one can think of \mathcal{A}_Υ as the configuration space for the graph Υ , that is homeomorphic to G^N .

What we are doing at the moment is to construct relevant configuration functions. In particular, what we need is to consider generalizations of the Wilson loops $T[\alpha]$ defined previously, making use of the graphs and the space \mathcal{A}_Υ . Every graph Υ can be decomposed into independent loops α_i and the corresponding Wilson loops $T[\alpha_i]$ are a particular example of functions defined over \mathcal{A}_Υ . What we shall consider as a generalization of the Wilson loop are *all* possible functions defined over \mathcal{A}_Υ . Thus, a function $c : G^N \rightarrow \mathbb{C}$ defines a *cylindrical* function C_Υ of the connection A as,

$$C_\Upsilon := c(h(e_1), h(e_2), \dots, h(e_N)) \quad (15)$$

By considering all possible functions c and all possible embedded graphs Υ , we generate the algebra of functions known as Cyl (it is closely related to the holonomy algebra, and it can be converted into a C^* -algebra $\overline{\text{Cyl}}$, by suitable completion).

Let us now discuss why this choice of configuration functions is compatible with the basic guiding principles for the quantization we are building up, namely diffeomorphism invariance and background independence. Background independence is clear since there is no need for a background metric to define the holonomies. Diffeomorphism invariance is a bit more subtle. Clearly, when one applies a diffeomorphism $\phi : \Sigma \rightarrow \Sigma$, the holonomies transform in a covariant way

$$\phi_* \cdot h(e_I) = h(\phi^{-1} \cdot e_I), \quad (16)$$

that is, the diffeomorphism acts by moving the edge (or loop). How can we then end up with a diffeo-invariant quantum theory? The strategy in LQG is to look for a *diffeomorphism invariant* representation of the diffeo-covariant configuration functions. As we shall see later, this has indeed been possible and in a sense represents the present ‘success’ of the approach.

Let us now consider the functions depending of the momenta that will be fundamental in the (loop) quantization. The basic idea is again to look for functions that are defined in a background independent way, that are natural from the view point of the geometric character of the object (1-form, 2-form, etc), and that transform covariantly with respect to the gauge invariances of the theory. Just as the the connection A_a^i can be identified with a one form that could be integrated along a one-dimensional object, one would like analyze the geometric character of the densitised triad \tilde{E} in order to naturally define a smeared object. Recall that the momentum is a density-one vector field on Σ , \tilde{E}_i^a with values in the dual of the lie-algebra $su(2)$. In terms of its tensorial character, it is naturally dual to a (lie-algebra valued) two form,

$$E_{ab\ i} := \frac{1}{2} \eta_{abc} \tilde{E}_i^c \quad (17)$$

where η_{abc} is the naturally defined Levi-Civita symbol. It is now obvious that the momenta is crying to be integrated over a two-surface S . It is now easy to define the objects

$$E[S, f] := \int_S E_{ab\ i} f^i dS^{ab}, \quad (18)$$

where f^i is a lie-algebra valued smearing function on S . This ‘Electric flux’ variable does not need a background metric to be defined, and it transforms again covariantly as was the case of the holonomies. The algebra generated by holonomies and flux variables is known as the *Holonomy-Flux* algebra \mathcal{HF} .

Perhaps the main reason why this Holonomy-Flux algebra \mathcal{HF} is interesting, is the way in which the basic generators interact, when considering the classical (Poisson) lie-bracket. First, given that the configuration functions depend only on the connection and the connections Poisson-commute, one expects that $\{T[\alpha], T[\beta]\} = 0$ for any loops α and β . The most interesting poisson bracket one is interested in is the one between a configuration and a momenta variable,

$$\{T[\alpha], E[S, f]\} = \kappa \sum_{\mu} f^i(v) \iota(\alpha, S|_v) \text{Tr}(\tau_i h(\alpha)) \quad (19)$$

where the sum is over the vertices v and $\iota(\alpha, S|_v) = \pm 1$ is something like the intersection number between the loops α and the surface S at point v . The sum is over all intersection of the loop α and the surface S . The most important property of the Poisson Bracket is that it is completely topological. This has to be so if we want to have a fully background independent classical algebra for the quantization.

A remark is in order. The value of the constant $\iota|_v$ depends not only on the relative orientation of the tangent vector of the loop α with respect to the orientation of Σ and S , but also on a further decomposition of the loop into edges, and whether they are ‘incoming’ or ‘outgoing’ to the vertex v . The end result is that is we have, for simple intersections, that the number $\iota|_v$ becomes insensitive to the ‘orientation’. This is different to the $U(1)$ case where the final result *is* the intersection number. For details see [12].

Let us now consider the slightly more involved case of a cylindrical function C_{Υ} that is defined over a graph Υ with edges e_I , intersecting the surface S at points p . We have then,

$$\{C_{\Upsilon}, E[S, f]\} = \frac{\kappa}{2} \sum_p \sum_{I_p} \iota(I_p) f^i(p) X_{I_p}^i \cdot c \quad (20)$$

where the sum is over the vertices p of the graph that lie on the surface S , I_p are the edges starting or finishing in p and where $X_{I_p}^i \cdot c$ is the result of the action of the i -th left (resp. right) invariant vector field on the I_p -th copy of the group if the I_p -th edge is pointing away from (resp. towards) the surface S . Note the structure of the right hand side. The result is non-zero only if the graph Υ used in the definition of the configuration variable C_{Υ} intersects the surface S used to smear the triad. If the two intersect, the contributions arise from the action of right/left invariant vector fields on the arguments of c associated with the edges at the intersection.

Finally, the next bracket we should consider is between two momentum functions, namely $\{E[S, f], E[S', g]\}$. Just as in the case of holonomies, these functions depend only on one of the canonical variables, namely the triad \tilde{E} . One should then expect that their Poisson bracket vanishes. Surprisingly, this is *not* the case and one has to appropriately define the correct algebraic structure¹.

¹ The end result is that one should not regard $E[S, f]$ as phase space functions subject to the ordinary Poisson bracket relations, but rather should be viewed as arising from vector fields X^{α} on \mathcal{A} . The non-trivial bracket is then due to the non-commutative nature of the corresponding vector fields. This was shown in [12] where details can be found

We have arrived then to the basic variables that will be used in the quantization in order to arrive at LQG. They are given by,

$$h(e_I) \quad \text{Configuration function} \quad (21)$$

and

$$E[S, f] \quad \text{Momentum function,} \quad (22)$$

subject to the basic Poisson bracket relations given by Eqs. (19) and (20). In the next section we shall take the Holonomy-Flux algebra \mathcal{HF} as the starting point for the quantization.

III. LOOP QUANTUM GEOMETRY

Let us now consider the particular representation that defines LQG. As we have discussed before, the basic observables are represented as operators acting on wave functions $\Psi_\Upsilon(\bar{A}) \in \mathcal{H}_{\text{kin}}$ as follows:

$$\hat{h}(e_I) \cdot \Psi(\bar{A}) = (h(e_I) \Psi)(\bar{A}) \quad (23)$$

and

$$\hat{E}[S, f] \cdot \Psi_\Upsilon(\bar{A}) = i\hbar \{\Psi_\Upsilon, {}^2E[S, f]\} = i8\pi \frac{\ell_P^2}{2} \sum_p \sum_{I_p} \kappa(I_p) f^i(p) X_{I_p}^i \cdot \psi \quad (24)$$

where $\ell_P^2 = G\hbar$, the Planck area is giving us the scale of the theory (recall that the Immirzi parameter γ does not appear in the basic Poisson bracket, and should therefore not play any role in the quantum representation). Here we have assumed that a cylindrical function Ψ_Υ on a graph Υ is an element of the Kinematical Hilbert space (which we haven't defined yet!). This implies one of the most important assumption in the loop quantization prescription, namely, that objects such as holonomies and Wilson loops that are smeared in one dimension are well defined operators on the quantum theory².

The basic idea for the construction of both the Hilbert space \mathcal{H}_{kin} (with its measure) and the quantum configuration space $\bar{\mathcal{A}}$, is to consider *all* possible graphs on Σ . For any given graph Υ , we have a configuration space $\mathcal{A}_\Upsilon = (SU(2))^N$, which is n -copies of the (compact) gauge group $SU(2)$. Now, it turns out that there is a preferred (normalized) measure on any compact semi-simple Lie group that is left and right invariant. It is known as the Haar measure μ_H on the group. We can thus endow \mathcal{A}_Υ with a measure μ_Υ that is defined by using the Haar measure on all copies of the group. Given this measure on \mathcal{A}_Υ , we can consider square integrable functions thereon and with them the graph- Υ Hilbert space \mathcal{H}_Υ , which is of the form:

$$\mathcal{H}_\Upsilon = L^2(\mathcal{A}_\Upsilon, d\mu_\Upsilon) \quad (25)$$

If we were working with a unique, fixed graph Υ_0 (which we are not), we would be in the case of a lattice gauge theory on an irregular lattice. The main difference between that situation and LQG is that, in the latter case, one is considering all graphs on Σ , and one has a family of configurations spaces $\{\mathcal{A}_\Upsilon / \Upsilon \text{ a graph in } \Sigma\}$, and a family of Hilbert spaces $\{\mathcal{H}_\Upsilon / \Upsilon \text{ a graph in } \Sigma\}$.

² this has to be contrasted with the ordinary Fock representation where such objects do *not* give raise to well defined operators on Fock space. This implies that the loop quantum theory is qualitatively different from the standard quantization of gauge fields.

The quantum configuration space $\overline{\mathcal{A}}$ is the configuration space for the “largest graph”; and similarly, the kinematical Hilbert space \mathcal{H}_{kin} is the largest space containing all Hilbert spaces in $\{\mathcal{H}_{\Upsilon} / \Upsilon \text{ a graph in } \Sigma\}$. The Ashtekar-Lewandowski measure μ_{AL} on \mathcal{H}_{kin} is then the measure whose projection to any \mathcal{A}_{Υ} yields the corresponding Haar measure μ_{Υ} . The resulting Hilbert space can thus be written as

$$\mathcal{H}_{\text{kin}} = L^2(\overline{\mathcal{A}}, d\mu_{\text{AL}})$$

The cylindrical functions $\Psi_{\Upsilon} \in \text{Cyl}$ belong to the Hilbert space of the theory.

Let us then recall what is the structure of simple states in the theory. The vacuum or ‘ground state’ $|0\rangle$ is given by the unit function. One can then create excitations by acting via multiplication with holonomies or Wilson loops. The resulting state $|\alpha\rangle = \hat{T}[\alpha] \cdot |0\rangle$ is an excitation of the geometry but only along the one dimensional loop α . Since the excitations are one dimensional, the geometry is sometimes said to be *polymer like*. In order to obtain a geometry that resembles a three dimensional continuum one needs a huge number of edges (10^{68}) and vertices.

A. A choice of basis: Spin Networks

The purpose of this part is to provide a useful decomposition of the Hilbert space \mathcal{H}_{Υ} , for all graphs. From our previous discussion we know that the Hilbert space \mathcal{H}_{Υ} is the Cauchy completion of the space of cylinder functions on Υ , Cyl_{Υ} with respect to the norm induced by the Haar measure on the graph configuration space $\mathcal{A}_{\Upsilon} = (\text{SU}(2))^N$. Thus, what we are looking for is a convenient basis for functions $F_{\Upsilon}(A)$ of the form

$$F_{\Upsilon}(A) := f(h(e_1), h(e_2), \dots, h(e_N)) \quad (26)$$

Let us for a moment consider just one edge, say e_i . What we need to do is to be able to decompose any function F on G (in this case we only have one copy of the group), in a suitable basis.

In the case of the group $G = \text{SU}(2)$, there is a decomposition of a function $f(g)$ of the group ($g \in G$). It reads,

$$f(g) = \sum_j \sqrt{j(j+1)} f_j^{mm'} \overset{j}{\Pi}_{mm'}(g) \quad (27)$$

where,

$$f_j^{mm'} = \sqrt{j(j+1)} \int_G d\mu_H \overset{j}{\Pi}_{mm'}(g^{-1}) f(g) \quad (28)$$

and is the equivalent of the Fourier component. The functions $\overset{j}{\Pi}_{mm'}(g)$ play the role of the Fourier basis. In this case these are unitary representation of the group, and the label j labels the irreducible representations. In the $\text{SU}(2)$ case with the interpretation of spin, these represent the spin- j representations of the group. In our case, we will continue to use that terminology (spin) even when the interpretation is somewhat different.

Given a cylindrical function $\Psi_{\Upsilon}[A] = \psi(h(e_1), h(e_2), \dots, h(e_N))$, we can then write an expansion for it as,

$$\begin{aligned} \Psi_{\Upsilon}[A] &= \psi(h(e_1), h(e_2), \dots, h(e_N)) \\ &= \sum_{j_1 \dots j_N} f_{j_1 \dots j_N}^{m_1 \dots m_N, n_1 \dots n_N} \phi_{m_1 n_1}^{j_1}(h(e_1)) \dots \phi_{m_N n_N}^{j_N}(h(e_N)), \end{aligned} \quad (29)$$

where $\phi_{mn}^j(g) = \sqrt{j(j+1)} \hat{\Pi}_{mn}^j(g)$ is the normalized function satisfying

$$\int_G d\mu_H \overline{\phi_{mn}^j(g)} \phi_{m'n'}^{j'}(g) = \delta_{j,j'} \delta_{m,m'} \delta_{n,n'}.$$

The expansion coefficients can be obtained by projecting the state $|\Psi_\Upsilon\rangle$,

$$f_{j_1 \dots j_N}^{m_1 \dots m_N, n_1 \dots n_N} = \langle \phi_{m_1 n_1}^{j_1} \dots \phi_{m_N n_N}^{j_N} | \Psi_\Upsilon \rangle \quad (30)$$

This implies that the products of components of irreducible representations $\prod_{i=1}^N \phi_{m_i n_i}^{j_i}[h(e_i)]$ associated with the N edges $e_I \in \Upsilon$, for all values of spins j and for $-j \leq m, n \leq j$ and for any graph Υ , is a complete orthonormal basis for \mathcal{H}_{kin} . We can then write,

$$\mathcal{H}_\Upsilon = \otimes_j \mathcal{H}_{\Upsilon,j} \quad (31)$$

where the Hilbert space $\mathcal{H}_{\alpha,j}$ for a single loop α , and a label j is the familiar $(2j+1)$ dimensional Hilbert space of a particle of ‘spin j ’. For a complete treatment see [14].

In the case of geometry with group $SU(2)$, the graphs with labelling $j_I = \mathbf{j}$ are known as spin networks. As the reader might have noticed, in the geometry case there are more labels than the spins for the edges. Normally these are associated to vertices and are known as intertwiners. This means that the Hilbert spaces $\mathcal{H}_{\Upsilon,\mathbf{j}}$ is finite dimensional. Its dimension being a measure of the extra freedom contained in the intertwiners. One could then introduce further labelling \mathbf{l} for the graph, so we can decompose the Hilbert space as

$$\mathcal{H}_\Upsilon = \otimes_j \mathcal{H}_{\Upsilon,\mathbf{j}} = \otimes_{\mathbf{j},\mathbf{l}} \mathcal{H}_{\Upsilon,\mathbf{j},\mathbf{l}} \quad (32)$$

where now the spaces $\mathcal{H}_{\Upsilon,\mathbf{j},\mathbf{l}}$ are one-dimensional. For more details see [13], [14] and [2]. With this convenient basis it is simple to consider geometrical operator. The most important one in the study of black holes is given by the flux and the area operators that we consider next.

B. Flux and Area operators

The operators $\hat{E}[S, f]$ corresponding to the electric flux observables, are in a sense the basic building blocks for constructing the quantum geometry. We have seen in Sec.III the action of this operators on cylindrical functions,

$$\hat{E}[S, f] \cdot \Psi_\Upsilon(\bar{A}) = -i \hbar \{ \Psi_\Upsilon, {}^2E[S, f] \} = -i 8\pi \frac{\ell_P^2}{2} \sum_p \sum_{I_p} \kappa(I_p) f^i(p) X_{I_p}^i \cdot \psi \quad (33)$$

Here the first sum is over the intersections of the surface S with the graph Υ , and the second sum is over all possible edges I_p that have the vertex v_p (in the intersection of S and the graph) as initial or final point. In the simplest case of a loop α , there are only simple intersections (meaning that there are two edges for each vertex), and in the simplest case of only one intersection between S and α we have one term in the first sum and two terms in the second (due to the fact that the loop α is seen as having a vertex at the intersection point). In this simplified case we have

$$\hat{E}[S, f] \cdot \Psi_\alpha(\bar{A}) = -i 8\pi \ell_P^2 f^i(p) X_{I_p}^i \cdot \psi \quad (34)$$

Note that the action of the operator is to ‘project’ the angular momentum in the direction given by f^i (in the internal space associated with the Lie algebra). As we shall see, this operator is in a sense fundamental the fundamental entity for constructing (gauge invariant) geometrical operators. For this, let us rewrite the action of the flux operator (33), dividing the edges that are above the surface S , as ‘up’ edges, and those that lie under the surface as ‘down’ edges.

$$\hat{E}[S, f] \cdot = 8\pi \frac{\ell_P^2}{2} \sum_p f^i(p) (\hat{J}_{i(u)}^p - \hat{J}_{i(d)}^p). \quad (35)$$

where the sum is over the vertices at the intersection of the graph and the surface, and where the ‘up’ operator $\hat{J}_{i(u)}^p = \hat{J}_i^{p,e_1} + \hat{J}_i^{p,e_2} + \dots + \hat{J}_i^{p,e_u}$ is the sum over all the up edges and the down operator $\hat{J}_{i(d)}^p$ is the corresponding one for the down edges.

The second simplest operator that can be constructed representing geometrical quantities of interest is the *area operator*, associated to surfaces S . The reason behind this is again the fact that the densitized triad is dual to a two form that is naturally integrated along a surface. The difference between the classical expression for the area and the flux variable is the fact that the area is a gauge invariant quantity. Let us first recall what the classical expression for the area function is, and then we will outline the regularization procedure to arrive at a well defined operator on the Hilbert space. The area $A[S]$ of a surface S is given by $A[S] = \int_S d^2x \sqrt{h}$, where h is the determinant of the induced metric h_{ab} on S . When the surface S can be parametrized by setting, say, $x^3 = 0$, then the expression for the area in terms of the densitized triad takes a simple form:

$$A[S] = \gamma \int_S d^2x \sqrt{\tilde{E}_i^3 \tilde{E}_j^3 k^{ij}} \quad (36)$$

where $k^{ij} = \delta^{ij}$ is the Killing-Cartan metric on the Lie algebra, and γ is the Barbero-Immirzi parameter (recall that the canonical conjugate to A is $\gamma \tilde{E}_i^a = \tilde{E}_i^a / \gamma$). Note that the functions is again smeared in two dimensions and that the quantity inside the square root is very much a square of the (local) flux. One expects from the experience with the flux operator, that the resulting operator will be a sum over the intersecting points p , so one should focus the attention to the vertex operator

$$\Delta_{S,\Upsilon,p} = - \left[(\hat{J}_{i(u)}^p - \hat{J}_{i(d)}^p) (\hat{J}_{j(u)}^p - \hat{J}_{j(d)}^p) \right] k^{ij}$$

with this, the area operator takes the form,

$$\hat{A}[S] = 8\pi \gamma \ell_P^2 \sum_p \sqrt{\widehat{\Delta_{S,\Upsilon,p}}} \quad (37)$$

We can now combine both the form of the vertex operator with Gauss’ law $(\hat{J}_{i(u)}^p + \hat{J}_{i(d)}^p) \cdot \Psi = 0$ to arrive at,

$$|(\hat{J}_{i(u)}^p - \hat{J}_{i(d)}^p)|^2 = |2(\hat{J}_{i(u)}^p)|^2 \quad (38)$$

where we are assuming that there are no tangential edges. The operator $\hat{J}_{i(u)}^p$ is an angular momentum operator, and therefore its square has eigenvalues equal to $j^u(j^u + 1)$ where j^u is the label for the total ‘up’ angular momentum. We can then write the form of the operator

$$\hat{A}[S] \cdot \mathcal{N}(\Upsilon, \vec{j}) = 8\pi \gamma \ell_P^2 \sum_{v \in V} \sqrt{|\hat{J}_{i(u)}^p|^2} \cdot \mathcal{N}(\Upsilon, \vec{j}) \quad (39)$$

With these conventions, in the case of simple intersections between the graph Υ and the surface S , the area operator takes the well known form:

$$\hat{A}[S] \cdot \mathcal{N}(\Upsilon, \vec{j}) = 8\pi \gamma \ell_P^2 \sum_{v \in V} \sqrt{j_v(j_v + 1)} \cdot \mathcal{N}(\Upsilon, \vec{j})$$

when acting on a *spin network* $\mathcal{N}(\gamma, \vec{j})$ defined over Υ and with labels \vec{j} on the edges (we have not used a label for the intertwiners). As we shall see when we consider the quantum theory of isolated horizons, the two operators considered here will play an important role not only in the geometry of the horizon but in the entropy counting.

IV. QUANTUM ISOLATED HORIZONS

Let us focus on the sector of the theory consisting of space-times which admit a type I isolated horizon Δ with a fixed area a_o as the internal boundary. Then Σ is asymptotically flat and has an internal boundary S , topologically a 2-sphere, the intersection of Σ with Δ . The type I isolated horizon boundary conditions require that i) Δ be null, ii) Non-expanding, iii) The field equations be satisfied there and iv) the intrinsic geometry on Δ be left invariant by the null vector ℓ^a generating Δ . For details see [15].

Introduce on S an internal, unit, radial vector field r^i (i.e. any isomorphism from the unit 2-sphere in the Lie algebra of $SU(2)$ to S). Then it turns out that *the intrinsic geometry of S is completely determined by the pull-back $\underline{A}^i r_i =: W$ to S of the (internal-radial component of the) connection A^i on Σ* [15]. Furthermore, W is in fact a spin-connection intrinsic to the 2-sphere S . Finally, the fact that S is (the intersection of Σ with) a type I isolated horizon is captured in a relation between the two canonically conjugate fields:

$$F : \hat{=} dW \hat{=} - \frac{2\pi\gamma}{a_o} \underline{\Sigma}^i r_i. \quad (40)$$

where $\underline{\Sigma}^i$ is the pull-back to S of the 2-forms $\Sigma_{ab}^i = \eta_{abc} E^{ic}$ on Σ . (Throughout, $\hat{=}$ will stand for equality restricted to Δ .) Thus, because of the isolated horizon boundary conditions, fields which would otherwise be independent are now related. In particular, the pull-backs to S of the canonically conjugate fields A^i , Σ^i are completely determined by the $U(1)$ connection W .

In absence of an internal boundary, the symplectic structure is given just by a volume integral [15]. In presence of the internal boundary under consideration, it now acquires a surface term [6]:

$$\Omega(\delta_1, \delta_2) = \frac{1}{8\pi G} \left[\int_M \text{Tr} (\delta_1 A \wedge \delta_2 \Sigma - \delta_2 A \wedge \delta_1 \Sigma) + \frac{a_o}{\gamma\pi} \oint_S \delta_1 W \wedge \delta_2 W \right], \quad (41)$$

where $\delta \equiv (\delta A, \delta \Sigma)$ denotes tangent vectors to the phase space Γ . Since W is essentially the only ‘free data’ on the horizon, it is not surprising that the surface term of the symplectic structure is expressible entirely in terms of W . However, it is interesting that the new surface term is precisely the symplectic structure of a $U(1)$ -Chern Simons theory. *The symplectic structures of the Maxwell, Yang-Mills, scalar and dilatonic fields do not acquire surface terms and, because of minimal coupling, do not feature in the gravitational symplectic structure either.* Conceptually, this is an important point: this, in essence, is the reason

why the black hole entropy depends just on the horizon area and not, in addition, on the matter charges [6].

One can systematically ‘quantize’ this sector of the phase space [6]. We can focus only on the gravitational field since the matter fields do not play a significant role. One begins with a Kinematic Hilbert space $\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_S$ where \mathcal{H}_V is the Hilbert space of states in the bulk as described before and \mathcal{H}_S is the Hilbert space of surface states. Expression (41) of the symplectic structure implies that \mathcal{H}_S should be the Hilbert space of states of a Chern-Simons theory on the punctured S , where the ‘level’, or the coupling constant, is given by:

$$k = \frac{a_o}{4\pi\gamma\ell_{\text{Pl}}^2} \quad (42)$$

A pre-quantization consistency requirement is that k be an integer [6].

Our next task is to encode in the quantum theory the fact that Δ is a type I horizon with area a_o . This is done by imposing the horizon boundary condition as an *operator equation*:

$$(1 \otimes \hat{F}) \Psi \hat{=} - \left(\frac{2\pi\gamma}{a_o} (\hat{\underline{\Sigma}} \cdot r) \otimes 1 \right) \Psi, \quad (43)$$

on admissible states Ψ in \mathcal{H} . Now, a general solution to (43) can be expanded out in a basis: $\Psi = \sum_n \Psi_V^{(n)} \otimes \Psi_S^{(n)}$, where $\Psi_V^{(n)}$ is an eigenvector of the ‘triad operator’ $-(2\pi\gamma/a_o) (\hat{\underline{\Sigma}} \cdot r)(x)$ on \mathcal{H}_V and $\Psi_S^{(n)}$ is an eigenvector of the ‘curvature operator’ $\hat{F}(x)$ on \mathcal{H}_S with *same eigenvalues*. Thus, the theory is non-trivial only if a sufficiently large number of eigenvalues of the two operators coincide. Since the two operators act on entirely different Hilbert spaces and are introduced quite independently of one another, this is a *very* non-trivial requirement.

Now, in the bulk Hilbert space \mathcal{H}_V , the eigenvalues of the ‘triad operator’ are given by [12]:

$$- \left(\frac{2\pi\gamma}{a_o} \right) \left(8\pi\ell_{\text{Pl}}^2 \sum_I m_I \delta^3(x, p_I) \eta_{ab} \right), \quad (44)$$

where m_I are half integers and η_{ab} is the natural, metric independent Levi-Civita density on S and p_I are points on S at which the polymer excitations of the bulk geometry in the state Ψ_V puncture S . A completely independent calculation [6], involving just the surface Hilbert space \mathcal{H}_S , yields the following eigenvalues of $\hat{F}(x)$:

$$\frac{2\pi}{k} \sum_I n_I \delta^3(x, p_I) \equiv 2\pi \frac{4\pi\gamma\ell_{\text{Pl}}^2}{a_o} \sum_I n_I \delta^3(x, p_I) \quad (45)$$

where n_I are integers modulo k . Thus, with the identification $-2m_I = n_I \bmod k$, the two sets of eigenvalues match exactly. There is a further requirement or constraint that the numbers m_I should satisfy, namely,

$$\sum_I m_I = 0 \quad (46)$$

This constraint is sometimes referred to as the *projection constraint*, given that the ‘total projection of the angular momentum’ is zero. Note that in the Chern-Simons theory the eigenvalues of $F(x)$ are dictated by the ‘level’ k and the isolated horizon boundary conditions tie it to the area parameter a_o just in the way required to obtain a coherent description of the geometry of the quantum horizon.

In the classical theory, the parameter a_o in the expression of the surface term of the symplectic structure (41) and in the boundary condition (40) is the horizon area. However in the *quantum theory*, a_o has simply been a parameter so far; we have not tied it to the *physical area of the horizon*. Therefore, in the entropy calculation, to capture the intended physical situation, one constructs a suitable ‘micro-canonical’ ensemble. This leads to the last essential technical step.

Let us begin by recalling that, in quantum geometry, the area eigenvalues are given by,

$$8\pi\gamma\ell_{\text{Pl}}^2 \sum_I \sqrt{j_I(j_I + 1)}.$$

We can therefore construct a micro-canonical ensemble by considering only that sub-space of the volume theory which, at the horizon, satisfies:

$$a_o - \delta \leq 8\pi\gamma\ell_{\text{Pl}}^2 \sum_I \sqrt{j_I(j_I + 1)} \leq a_o + \delta \quad (47)$$

where I ranges over the number of punctures, j_I is the spin label associated with the puncture p_I [6].³

Quantum Einstein’s equations can be imposed as follows. The implementation of the Gauss and the diffeomorphism constraints is the same as in [6]. The first says that the ‘total’ state in \mathcal{H} be invariant under the $\text{SU}(2)$ gauge rotations of triads and, as in [6], this condition is automatically met when the state satisfies the quantum boundary condition (43). The second constraint says that two states are physically the same if they are related by a diffeomorphism. The detailed implementation of this condition is rather subtle because an extra structure is needed in the construction of the surface Hilbert space and the effect of diffeomorphisms on this structure has to be handled carefully [6]. However, the final result is rather simple: For surface states, what matters is only the number of punctures; their location is irrelevant. The last quantum constraint is the Hamiltonian one. In the classical theory, the constraint is differentiable on the phase space only if the lapse goes to zero on the boundary. Therefore, this constraint restricts only the volume states. However, there is an indirect restriction on surface states which arises as follows. Consider a set (p_I, j_I) with $I = 1, 2, \dots, N$ consisting of N punctures p_I and half-integers j_I , real, satisfying (47). We will refer to this set as ‘surface data’. Suppose there exists a bulk state satisfying the Hamiltonian constraint which is compatible with this ‘surface data’. Then, we can find compatible surface states such that the resulting states in the total Hilbert space \mathcal{H} lie in our ensemble. The space $S_{(p_I, j_I)}$ of these surface states is determined entirely by the surface data. In our state counting, we include the number $\mathcal{N}_{(p_I, j_I)}$ of these surface states, subject however, to the projection constraint that is purely intrinsic to the horizon.⁴

³ The appearance of the parameter δ is standard in statistical mechanics. It has to be much smaller than the macroscopic parameters of the system but larger than level spacings in the spectrum of the operator under consideration.

⁴ Note that there may be a large number –possibly infinite– of bulk states which are compatible with a given ‘surface data’ in this sense. This number does not matter because the bulk states are ‘traced out’ in calculating the entropy of the horizon. What matters for the entropy calculation is only the dimensionality of $S_{(p_I, j_I)}$.

V. BLACK HOLE ENTROPY

In this section we shall deal with the issue of entropy counting. We have started with a type I isolated horizon of area a_o (in vacuum this is the only multipole defining the horizon), and we have quantized the theory and arrived to a Hilbert space as described before. The question now is: How many microstates correspond to the given macrostate, defined uniquely by a_o ?

Let us now pose the condition that the states in $S_{(p_I, j_I)}$ should satisfy:

- They belong to the *physical* Hilbert space on the surface \mathcal{H}_S .
- The condition (47) is satisfied.
- The quantum boundary condition (45) is satisfied.
- The projection constraint (46) is satisfied.

In terms of a concrete counting the problem is posed as follows: We shall consider the lists (p_I, j_I, m_I) corresponding to the allowed punctures, spins of the piercing edges, and ‘projected angular momentum’ labels, respectively.

The task is then to count these states and find $\mathcal{N}_{(p_I, j_I, m_I)}$. The entropy will be then,

$$S_{\text{BH}} := \ln(\mathcal{N}_{(p_I, j_I, m_I)}) \quad (48)$$

This problem was systematically addressed in [6] in the approximation of *large* horizon area a_o . Unfortunately, the number of such states was underestimated in [6].⁵ In [9] the counting was completed and it was shown in detail that, for large black holes (in Planck units), the entropy behaves as:

$$S_{\text{BH}} = \frac{A}{4} - \frac{1}{2} \ln A,$$

provided the Barbero-Immirzi parameter γ is chosen to coincide with the value γ_0 , that has to satisfy [9]:

$$1 = \sum_I (2j_I + 1) \exp \left(-2\pi\gamma_0 \sqrt{j_I(j_I + 1)} \right). \quad (49)$$

The solution to this transcendental equation is approximately $\gamma_0 = 0.27398 \dots$ (see [9, 17] for details).

Here we shall perform the counting in a different regime, namely for small black holes in the Planck regime [18]. Thus we shall perform no approximations as in the previous results. Thus our results are complementary to the analytic treatments. On the other hand our counting will be exact, since the computer algorithm is designed to count all states allowed. Counting configurations for large values of the area (or mass) is extremely difficult for the simple reason that the number of states scales exponentially. Thus, for the computing power at our disposal, we have been able to compute states up to a value of area of about $a_o = 550 \, l_P^2$ (recall that the minimum area gap for a spin 1/2 edge is about $a_{\min} \approx 6 \, l_P^2$, so the number of punctures on the horizon is below 100). At this point the number of

⁵ The under-counting was noticed in [16], and a new counting was there proposed and carried out in [8]. However the choice of relevant states to be counted there is slightly different from our case. Details of the comparison between two methods will be reported elsewhere [17].

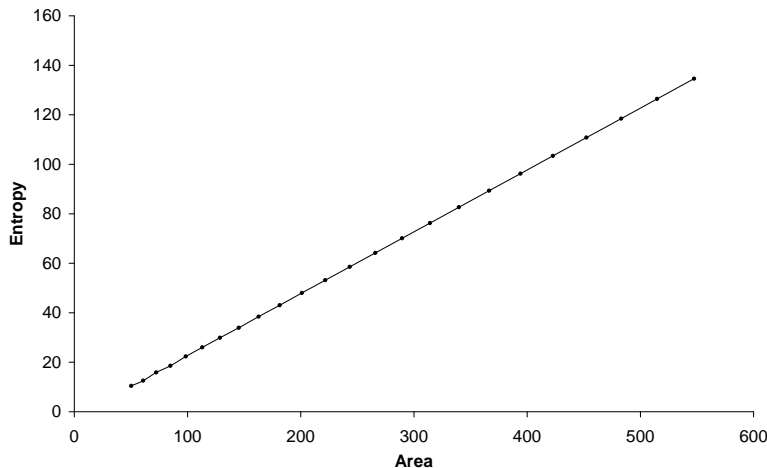


FIG. 1: The entropy as a function of area is shown, where the projection constraint has not been imposed. The BI is taken as $\gamma = 0.274$.

states exceeds 2.8×10^{58} . In terms of Planck masses, the largest value we have calculated is $M = 3.3 M_P$. When the projection constraint is introduced, the upper mass we can calculate is much smaller, given the computational complexity of implementing the condition. In this case, the maximum mass is about $1.7 M_P$.

It is important to describe briefly what the program for counting does. What we are using is what it is known, within combinatorial problems, as a brute force algorithm. This is, we are simply asking the computer to perform all possible combinations of the labels we need to consider, attending to the distinguishability -indistinguishability criteria that are relevant [6, 17], and to select (count) only those that satisfy the conditions needed to be considered as permissible combinations, i.e., the area condition and the spin projection constraint. An algorithm of this kind has an important disadvantage: it is obviously not the most optimized way of counting and the running time increases rapidly as we go to little higher areas. This is currently the main limitation of our algorithm. But, on the other hand, this algorithm presents a very important advantage, and this is the reason why we are using it: its explicit counting guarantees us that, if the labels considered are correct, the result must be the right one, as no additional assumption or approximation is being made. It is also important to have a clear understanding that the algorithm does not rely on any particular analytical counting available. That is, the program counts states as specified in the original formalism [6]. The computer program has three inputs: i) the classical mass M (or area $a_o = 16\pi M^2$), ii) The value of γ and iii) The size of the interval δ .

Once these values are given, the algorithm computes the level of the horizon Chern Simons theory $k = [a_o/4\pi\gamma]$ and the maximum number of punctures possible $n_{\max} = [a_o/4\pi\gamma\sqrt{3}]$ (where $[\cdot]$, indicates the principal integer value). At first sight we see that the two conditions we have to impose to permissible combinations act on different labels. The area condition acts over j 's and the spin projection constraint over m 's. This allows us to first perform combinations of j 's and select those satisfying the area condition. After that, we can perform combinations of m 's only for those combinations of j 's with the correct area, avoiding some

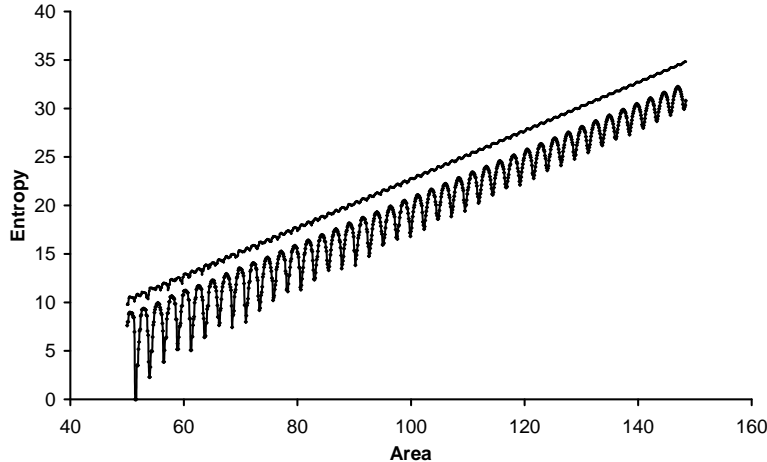


FIG. 2: Entropy *vs* Area with and without the projection constraint, with $\delta = 0.5$.

unnecessary work. We could also be allowed to perform the counting without imposing the spin projection constraint, by simply counting combinations of j 's and including a multiplicity factor of $\prod_I (2j_I + 1)$ for each one, accounting for all the possible combinations of m 's compatible with each combination of j 's. This would reduce considerably the running time of the program, as no counting over m 's has to be done, and will allow us to separate the effects of the spin projection constraint (that, as we will see, is the responsible of a logarithmic correction). It is very important to notice at this point that this separation of the counting is completely compatible with the distinguishability criteria.

The next step of the algorithm is to consider, in increasing order, all the possible number of punctures (from 1 to n_{\max}) and in each case it considers all possible values of j_I . Given a configuration (j_1, j_2, \dots, j_n) ($n \leq n_{\max}$), we ask whether the quantum area eigenvalue $A = \sum_I 8\pi\gamma\sqrt{j_I(j_I + 1)}$ lies within $[a_o - \delta, a_o + \delta]$. If it is not, then we go to the next configuration. If the answer is positive, then the labels m 's are considered as described before. That is, for each of them it is checked whether $\sum m_I = 0$ is satisfied.

In Figure 1, we have plotted the entropy, as $S = \ln(\# \text{states})$ *vs* the area a_o , where we have counted all possible states without imposing the $\sum m_I$ constraint, and have chosen a $\delta = 0.5$. As it can be seen, the relation is amazingly linear, even for such small values of the area. When we fix the BI parameter to be $\gamma = \gamma_0 = 0.274$, and approximate the curve by a linear function, we find that the best fit is for a slope equal to 0.2502.

When we include the projection constraint, the computation becomes more involved and we are forced to consider a smaller range of values for the area of the black holes. In Figure 2, we plot both the entropy without the projection and with the projection, keeping the same δ . The first thing to note is that for the computation with the constraint implemented, there are some large oscillations in the number of states. Fitting a straight line gives a slope of 0.237. In order to reduce the oscillations, we increased the size of δ to $\delta = 2$. The result is plotted in Figure 3. As can be seen the oscillations are much smaller, and the result of implementing the constraint is to shift the curve down (the slope is now 0.241). In order to compare it with the expected behavior of $S = A/4 - (1/2) \ln A$, we subtracted both curves

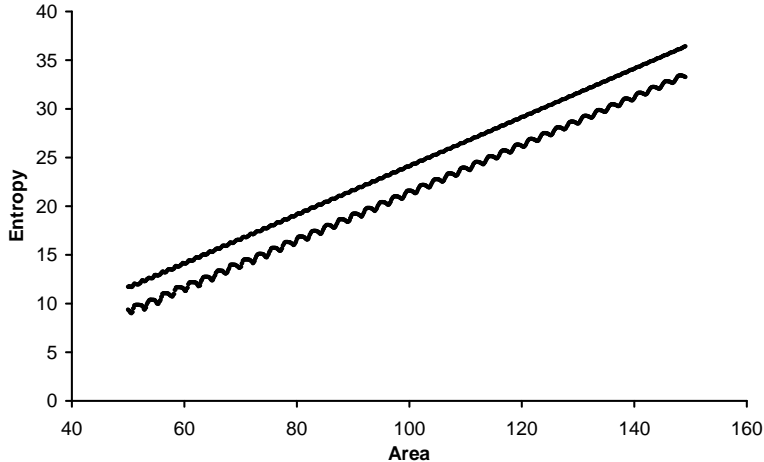


FIG. 3: Entropy *vs* Area with and without the projection constraint, with $\delta = 2$.

of Figure 3, in the range $a_o = [50, 160]$, and compared the difference with a logarithmic function. The coefficient that gave the best fit is equal to -0.4831 (See Figure 4). What can we conclude from this? While it is true that the rapidly oscillating function is far from the analytic curve, it is quite interesting that the oscillatory function follows a logarithmic curve with the “right” coefficient. It is still a challenge to understand the meaning of the oscillatory phase. Even when not conclusive by any means, we can say that the counting of states is consistent with a (n asymptotic) logarithmic correction with a coefficient equal to $(-1/2)$.

VI. DISCUSSION

In this contribution we have considered the approach to the quantum theory of gravity known as loop quantum gravity. We have presented a brief introduction to the main ideas behind this approach and have considered one of its main applications, black hole entropy. We have discussed the main features in the approach to black hole entropy, in particular in the implementation of the isolated horizon boundary conditions to the quantum theory and how this conditions tell us what states can be regarded as ‘black hole states’ that contribute to the entropy of the horizon. As we have seen, the fact that there is an intrinsic discreteness in the quantum horizon theory and that we are ignoring (tracing out) the states in the bulk, is the reason why the entropy becomes finite. It is sometimes believed that the fact that we do get an entropy proportional to area is natural are not surprising, given that on the horizon, the theory under consideration is a Chern-Simons theory with punctures, and the entropy of a two dimensional theory should be proportional to the total volume (area in this case). It is important to stress that the result is not as trivial as it sounds. To begin with, we do *not* have a given Chern-Simons theory on the horizon, for any macro-state of a given area a_o , there are many possible microstates that can be associated with it. They do *not* all live on the same ‘Chern-Simons Hilbert space’. The surface Hilbert space \mathcal{H}_S is made out

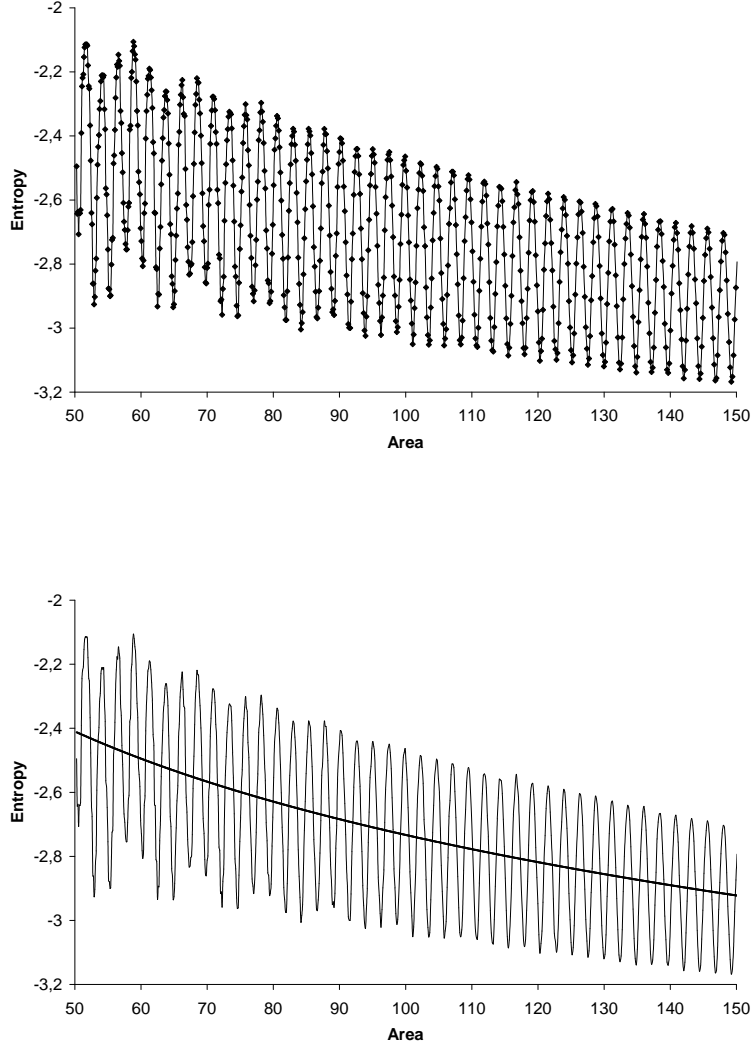


FIG. 4: The curves of Fig. 3 are subtracted and the difference, an oscillatory function, shown in the upper figure. The curve is approximated by a logarithm curve in the lower figure.

of the tensor product of all possible Chern-Simons states compatible with the constraints detailed in Sec. refsec:5, which belong to different Chern-Simons states (characterized by, say, the total number of punctures). That the total number of states compatible with the (externally imposed) constraints is proportional to area is thus a rather non-trivial result.

One might also wonder about the nature of the entropy one is associating to the horizon. there has been some controversy about the origin and location of the degrees of freedom responsible for black hole entropy (see for instance [19] for a recent discussion). It has been argued that the degrees of freedom lie behind the horizon, on the horizon and even on an asymptotic region at infinity. What is then our viewpoint on this issue? The viewpoint is that the IH boundary conditions implement in a consistent manner an effective description, as horizon data, of the degrees of freedom that might have formed the horizon. These

degrees of freedom, even if they are there in the physical space-time, they are inaccessible to an external observer. The only thing that this observer can ‘see’ are the degrees of freedom at the horizon, and these degrees of freedom are thus responsible for the entropy associated to the horizon.

In the last part of this article, we have focussed our attention on some recent results pertaining to the counting of states for Planck size horizons. As we have shown, even when these black holes lie outside the original domain of validity of the isolated horizon formalism (tailored for large black holes), the counting of such states has shed some light on such important issues as the BI parameter, responsible for the asymptotic behavior, and the first order, logarithmic, correction to entropy. We have also found, furthermore, that there is a rich structure underlying the area spectrum and the number of black hole states that could not have been anticipated by only looking at the large area limit. In particular, it has been found that the apparent periodicity in the entropy *vs* area relation yields an approximate ‘quantization of the entropy that makes contact with Bekenstein’s heuristic considerations [20], and is independent on the choice of relevant states, and its associated counting. Details will be published elsewhere [21].

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